MAT 219 (lecture 8)

§7.5: Distinct Eigenvalues
→ More examples / Phase Portraits

§7.6: Complex Eigenvalues

Next: §7.7: Fundamental Matrices &

§7.2-3: Fill in some Linear Algebra theory (finally!)

Last time: New way to write answers "Fundamental Matrix" Ψ

→ Three ways to write solutions to DE:

EX: $\begin{aligned} x' &= 3x + 2y \\ y' &= x + 2y \end{aligned} \rightsquigarrow \underline{x}' = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \underline{x}$

Eigenvalues $\lambda = 1, 4$

Eigenvectors $\lambda = 1 \rightsquigarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 4 \rightsquigarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Check: $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \overset{?}{=} 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \overset{?}{=} 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Solutions:

① $\underline{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}$

split apart.

② $\begin{bmatrix} x = c_1 e^{t} + 2c_2 e^{4t} \\ y = -c_1 e^{t} + c_2 e^{4t} \end{bmatrix}$

combine.

③ $\underline{x} = \underbrace{\begin{bmatrix} e^{t} & 2e^{4t} \\ -e^{t} & e^{4t} \end{bmatrix}}_{\Psi} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Why Ψ?

Use it to solve for c_1 & c_2 in I.V.P.

EX: $\underline{x}' = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \underline{x}$ with $\underline{x}(0) = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

...

Gen soln: $\underline{x} = \begin{bmatrix} e^{t} & 2e^{4t} \\ -e^{t} & e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Initial Values:

$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \underline{x}(0) = \underbrace{\begin{bmatrix} \cancel{e^{0}}\,1 & 2\cancel{e^{4\cdot 0}}\,1 \\ -\cancel{e^{0}}\,1 & \cancel{e^{4\cdot 0}}\,1 \end{bmatrix}}_{\text{inverse!}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \left(\frac{1}{1-(-2)} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} 4 - 10 \\ 4 + 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$$\boxed{\underline{x} = -2 \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} + 3 \begin{bmatrix} 2e^{4t} \\ -e^{4t} \end{bmatrix}}$$

$$\boxed{\text{Recall: } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

EX: $\underline{x}' = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \underline{x}$ with $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solve & mark solution on phase plane.

Eigenvalues

$$\det \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 4)(\lambda - 2) = 0$$

$$\boxed{\lambda = 2, 4.}$$

Eigenvectors

$\lambda = 2$: $\begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix} \underline{v} = \underline{0}$

$$v = \begin{bmatrix} -(-1) \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

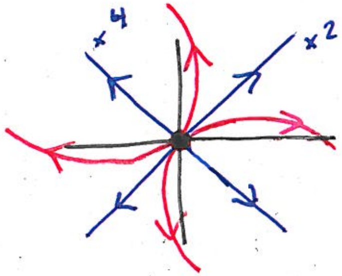
$\lambda = 4$: $\begin{bmatrix} 3-4 & -1 \\ -1 & 3-4 \end{bmatrix} \underline{v} = \underline{0}$

$$\underline{v} = \begin{bmatrix} -(-1) \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

(General Fact: Symm. Matrices have $\perp$ eigenvect)

General Solution

$$\underline{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}$$



$$\underline{x} = \underbrace{\begin{bmatrix} e^{2t} & e^{4t} \\ e^{2t} & -e^{4t} \end{bmatrix}}_{\text{Fundamental Matrix}} \underline{c}$$

"Unstable Node" or "Node Source"

(Solutions bend towards "louder" eigenvector)

(EX continues)

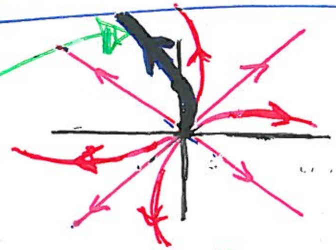
Initial Values

$$\begin{bmatrix} e^0 & e^0 \\ e^0 & -e^0 \end{bmatrix} \underline{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

$$\Rightarrow \underline{c} = \frac{1}{-1-1} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\boxed{\underline{x} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}}$$



solution through $\begin{bmatrix} x = 0 \\ y = 1 \end{bmatrix}$

Note: $x \to -\infty$
$y \to \infty$

(Check: $x = \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t} \xrightarrow{\checkmark} -\infty$ as $t \to \infty$

$y = \frac{1}{2}e^{2t} + \frac{1}{2}e^{4t} \xrightarrow{\checkmark} \infty$ as $t \to \infty$)

Drawing Phase Portraits

3 Basic Types of Picture (so far)

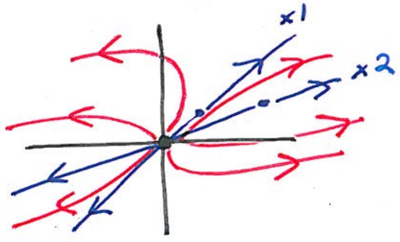
- Eigenvalues have same sign : "Node" → "Source" (+), → "Sink" (−)

EX $\underline{x}' = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \underline{x}$

$\lambda = 1, 2$

$\lambda = 1 : \underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = 2 : \underline{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$



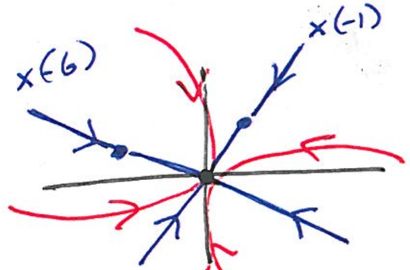
$\underline{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t}$

EX $\underline{x}' = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \underline{x}$

$\lambda = -1, -6$

$\lambda = -1 : \underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -6 : \underline{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$



$\underline{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-6t}$

(Solutions curve towards "louder" eigenvector)

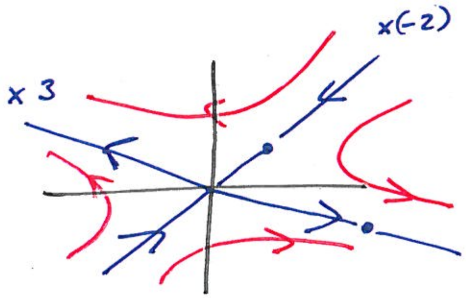
- Eigenvalues have different sign : "Saddle"

EX: $\underline{x}' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \underline{x}$

$\lambda = -2, 3$

$\lambda = -2 : \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 3 : \underline{v} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$



$\underline{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{3t}$

(Solutions "hug" "quieter" eigenvector)

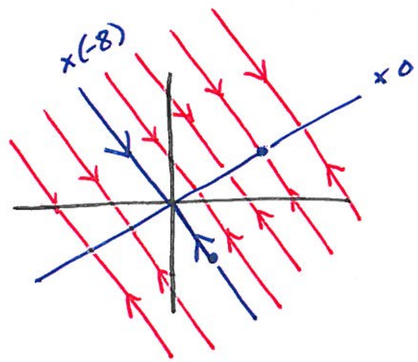
- One eigenvalue is zero: "Lines"

EX: $\underline{x}' = \begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix} \underline{x}$

$\lambda = -8, 0$

$\lambda = -8 : \underline{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = 0 : \underline{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

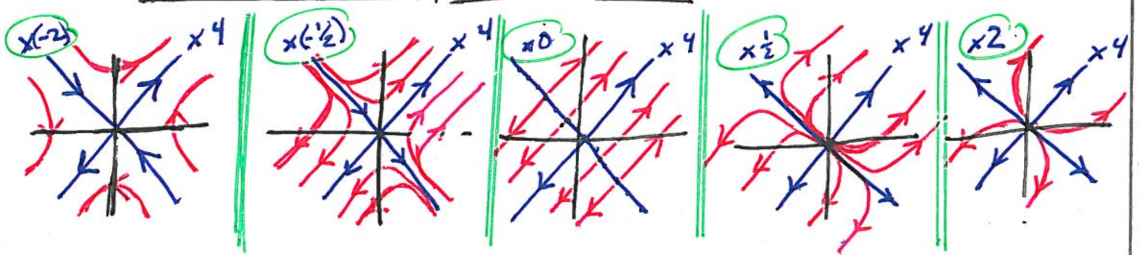


$\underline{x} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-8t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

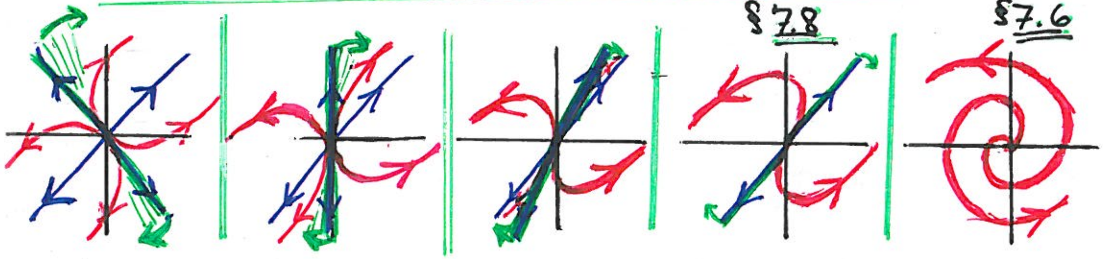
$e^{0t} = 1$

§7.6 Complex Eigenvalues (First, look at phase portraits)

Evolution of phase planes



Moving Eigenvalues



Moving Eigenvectors

Recall: In MAT 120 we used parametric circles and spirals:

$$\begin{bmatrix} x(t) = \cos t \\ y(t) = \sin t \end{bmatrix} \text{ circle}$$

$$\begin{bmatrix} x(t) = t\cos t \\ y(t) = t\sin t \end{bmatrix} \text{ spiral}$$

→ Expect answers to have $\cos t$ & $\sin t$ instead of e^t

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Let's look at an example analytically:

EX: $\underline{x}' = \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix} \underline{x}$

Eigenvalues

$$\det \begin{bmatrix} 3-\lambda & 2 \\ -4 & -1-\lambda \end{bmatrix} = 0$$

Quadratic Formula
$ax^2 + bx + c = 0$
$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2}$$

$$\boxed{\lambda = 1 \pm 2i}$$

Alt: Complete Square
$(\lambda-1)^2 + 4 = 0$
$(\lambda-1)^2 = -4$
$\lambda - 1 = 2i$
$\lambda = 1 \pm 2i$

Note: Eigenvalues are conjugate.

Eigenvectors

$\lambda = 1 + 2i$

$$\begin{bmatrix} 3-(1+2i) & 2 \\ -4 & -1-(1+2i) \end{bmatrix} \underline{v} = \underline{0}$$

$$\begin{bmatrix} 2-2i & 2 \\ -4 & -2+2i \end{bmatrix} \underline{v} = 0 \qquad \underline{v} = \begin{bmatrix} 2 \\ -(2-2i) \end{bmatrix} \xrightarrow{x(\frac{1}{2})} \boxed{\begin{bmatrix} 1 \\ -1+i \end{bmatrix}}$$

check: $\begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1+i \end{bmatrix} = \begin{bmatrix} 3-2+2i \\ -4+1-i \end{bmatrix}$ OK

$(1+2i) \begin{bmatrix} 1 \\ -1+i \end{bmatrix} = \begin{bmatrix} 1+2i \\ -1-2-2i+i \end{bmatrix}$

(Recall: $(a+bi)(\alpha+\beta i) = (a\alpha - b\beta) + (a\beta + b\alpha)i$)

(EX continues)

$\lambda = 1-2i$

$$\begin{bmatrix} 3-(1-2i) & 2 \\ -4 & -1-(1-2i) \end{bmatrix} \underline{v} = \underline{0}$$

$$\begin{bmatrix} 2+2i & 2 \\ -4 & -2+2i \end{bmatrix} \underline{v} = \underline{0} \qquad \underline{v} = \begin{bmatrix} 2 \\ -(2+2i) \end{bmatrix} \xrightarrow{\times\left(\frac{1}{2}\right)} \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$$

(Note: Conjugate of other eigenvector.)

FACT: For a real matrix, (conjugate eigenvectors) $\Longrightarrow$ (conjugate eigenvalues)

General solution ??

$$\underline{x} = c_1 \begin{bmatrix} 1 \\ -1+i \end{bmatrix} e^{(1+2i)t} + c_2 \begin{bmatrix} 1 \\ -1-i \end{bmatrix} e^{(1-2i)t}$$

WTF? — Note: c_1 & c_2 are complex #'s (In fact, they are conjugates)

Use Euler's Formula:

$$e^{(1+2i)t} = e^t \cdot e^{2ti}$$
$$= e^t \cdot (\cos 2t + i \sin 2t)$$

(EX continues)

$$\underline{x} = c_1 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i \right) \left(e^t \cos 2t + i e^t \sin 2t \right)$$

$$+ c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} i \right) \left(\underbrace{e^t \cos(-2t)}_{e^t \cos(2t)} + i \underbrace{e^t \sin(-2t)}_{-e^t \sin(2t)} \right)$$

(Recall: $(a+bi)(\alpha+\beta i) = (a\alpha - b\beta) + (a\beta + b\alpha)i$)

$$= c_1 \left[\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t \cos 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \sin 2t \right) + \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t \sin 2t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \cos 2t \right) i \right]$$

$$+ c_2 \left[\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t \cos 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \sin 2t \right) + \left(-\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t \sin 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \cos 2t \right) i \right]$$

$$= \overbrace{\underbrace{(c_1 + c_2)}_{\text{Real \#s}}}^{\text{New } c_1} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t \cos 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \sin 2t \right) + \underbrace{\overbrace{(c_1 - c_2) i}^{\text{Real \#s}}}_{\text{New } c_2} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t \sin 2t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \cos 2t \right)$$

$$\boxed{\underline{x} = c_1 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t \cos 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \sin 2t \right) + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t \sin 2t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \cos 2t \right)}$$